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H.J.J. TE RIELE & Ph. SCHROEVERS

A COMPARATIVE SURVEY OF NUMERICAL METHODS FOR THE LINEAR GENERALIZED ABEL INTEGRAL EQUATION

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A comparative survey of numerical methods for the linear generalized Abel integral equation*)

by

H.J.J. te Riele & Ph. Schroevers

ABSTRACT

A numerical comparison is made between a number of important representatives of the following classes of methods: (i) collocation-, (ii) product integration- and (iii) global methods. Special attention is paid to the performance of these methods for problems with a non-smooth solution.

It turned out that, when only relatively low accuracy is required, a good choice would be a second or third order collocation method of Branca.

A new collocation method which accounts for possible non-smoothness of the solution near the origin, turned out to be advantageous when high accuracy is required, *both* for problems with a non-smooth solution, *and* for problems with a smooth solution.

KEY WORDS & PHRASES: *Abel integral equation, collocation method, product integration method, Chebyshev approximation*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

The linear generalized Abel integral equation we consider has the form:

$$(1.1)\qquad \int_0^x \frac{K(x,t)f(t)}{(x-t)^{\frac{1}{2}}}\,dt = g(x), \quad 0 \le x \le X < \infty,$$

where $g(x)$ and $K(x,t)$ are known functions.

As the performance of the methods to be presented is influenced by the smoothness of the solution $f(x)$ of (1.1), it may be convenient to have a priori knowledge of the behaviour of $f(x)$. The following special version of an existence and smoothness theorem by Atkinson [1] can then be used.

THEOREM 1.1. *Let* $g(x)$ *have the form*

$$(1.2)\qquad g(x) = x^{\beta}\tilde{g}(x), \quad 0 < x \le X, \quad \tilde{g} \in C^{n+1}[0,X],$$

for some integer $n \ge 0$ *and* $\beta > -\frac{1}{2}$. *Assume* $K(x,t)$ *is* $n+2$ *times continuously differentiable for* $0 \le t \le x \le X$ *and*

$$(1.3)\qquad K(x,x) \ne 0, \quad 0 \le x \le X.$$

Then there is a unique solution $f(x)$ *of (1.1) of the form:*

$$(1.4)\qquad f(x) = x^{\beta-\frac{1}{2}}[a + xL(x)], \quad x > 0$$

with $L(x) \in C^n[0,X]$. *The constant* $a = 0$ *if and only if* $\tilde{g}(0) = 0$. □

The most important numerical methods to solve (1.1) can be divided into three groups: (i) collocation methods, (ii) product integration methods and (iii) global methods. Characteristic for both collocation- and product integration algorithms is that we introduce a grid $\{x_i = i.h, i = 0,\dots,N, N := X/h\}$, with grid spacing (or step) h. We then calculate approximations to $f(x)$ on each interval $[x_i, x_{i+1}]$, $i = 0,\dots,N-1$, *successively*. Their difference lies in the fact that, in the case of product integration, an approximation to $K(x,t)\cdot f(t)$ is made on each interval: the resulting integrals can be computed analytically. In the case of collocation, however, an approximation to $f(x)$ itself is made on each interval and, in general, the resulting integrals have to be calculated numerically. A global method approximates $f(x)$ on the whole interval $[0,X]$ by a certain linear combination of basis functions. The coefficients of this approximation are calculated *simultaneously*.

In this paper we make a numerical comparison between

(i) collocation methods developed by Branca [2] and a new method by the authors,

(ii) product integration methods by Anderssen, de Hoog and Weiss [3],

(iii) a global method of Chawla and Kumar [4].

The test problems will be divided into three groups:

(i) smooth problems (i.e. problems for which the solution is sufficiently often differentiable),

(ii) non-smooth problems (i.e., solutions of the form $f(x) = \psi(x)+x^{\frac{1}{2}}\chi(x)$, $\chi(x)$ and $\psi(x)$ sufficiently smooth),

(iii) problems with strongly oscillating or rapidly decreasing solutions.

In the next paragraphs we will discuss each method in more detail. We will frequently make use of the following manipulations on (1.1):

(i) introduce the grid

$$\{x_i := ih, \quad i = 0,\dots,N, \quad h = X/N\} \tag{1.5}$$

for some $N \in \mathbb{N}$;

(ii) write (1.1) in the form

$$\sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} \frac{K(x,t)\ f(t)}{(x-t)^{\frac{1}{2}}}\, dt + \int_{x_k}^{x} \frac{K(x,t)\ f(t)}{(x-t)^{\frac{1}{2}}}\, dt = g(x), \tag{1.6}$$

$x \in (x_k, x_{k+1}]$ for some k with $0 \le k \le N-1$.

2. BRANCA'S METHODS

Branca [2] developed a second and third order method to which we will refer as BR2 and BR3, respectively. For both methods we introduce the grid (1.5).

a. BR2

f(x) is approximated by a continuous function which is a first-degree polynomial on each interval $[x_i, x_{i+1}] =: \sigma_i$, i.e.:

(2.1) $$f(x)\big|_{x\in\sigma_i} \approx P_i(x) := \frac{1}{h}[(x_{i+1}-x)f_i + (x-x_i)f_{i+1}], \quad i = 1,\dots,N;$$

f_i denotes an approximation to $f(x_i)$.

We write (1.1) in the form (1.6), substitute (2.1) and restrict the continuous variable x to the discrete set $\{ih,\ i = 1,\dots,N\}$. After an obvious change of variable we get the scheme:

(2.2) $$g(x_k) = h^{\frac{1}{2}} \sum_{j=0}^{k-1} \int_0^1 \frac{K(x_k,(j+\tau)h)[(1-\tau)f_j+\tau f_{j+1}]}{(k-j-\tau)^{\frac{1}{2}}}\, d\tau,$$

$$k = 1,\dots,N.$$

The integrals in (2.2) are calculated using 1 - point weighted Gauss quadrature with weight functions $(\ell-\tau)^{-\frac{1}{2}}$:

(2.3) $$\int_0^1 \frac{G(\tau)}{(\ell-\tau)^{\frac{1}{2}}}\, d\tau = \omega_\ell G(a_\ell) + R_\ell[G(\tau)], \quad \ell = 1,\dots,N.$$

The weights ω_ℓ and abscissae a_ℓ are determined by requiring:

(2.4) $$R_\ell[G(\tau)] = 0 \quad \text{for} \quad G(\tau) = \tau^i, \quad i = 0,1.$$

By using (2.3) and solving (2.2) for f_k, we get the scheme (writing g_k for $g(x_k)$):

(2.5) $$f_k = [\omega_1 a_1 K(x_k,(k-1+a_1)h)]^{-1}\cdot$$
$$\cdot\Big[h^{-\frac{1}{2}} g_k - \sum_{j=0}^{k-2} \omega_{k-j} K(x_k,(j+a_{k-j})h)[(1-a_{k-j})f_j + a_{k-j}f_{j+1}]$$
$$- \omega_1(1-a_1)f_{k-1}K(x_k,(k-1+a_1)h)\Big],$$
$$k = 1,\dots,N.$$

The required starting value f_0 can be calculated from

$$(2.6)\qquad f_0 = \lim_{x\to 0} \frac{g(x)}{2x^{\frac{1}{2}}K(0,0)} \qquad \text{(see e.g. [2], p.310).}$$

b. BR3

A third order method might be derived by approximating $f(x)$ by a second-degree polynomial on the intervals $[x_i,x_{i+2}]$, $i = 0,2,4,\dots,N/2$, taking N even. This would require the solution of a (2×2)-system in each step. To avoid this, Branca calculates such an approximation to $f(x)$ only on the interval $[x_0,x_2]$, thus finding f_1 and f_2 (f_0 is given, e.g. by (2.6)) and then calculates f_3 by approximating $f(x)$ on $[x_2,x_3]$ by a second-degree polynomial through the points (x_1,f_1), (x_2,f_2), (x_3,f_3). In general, he calculates f_n by putting a second degree polynomial through (x_{n-2},f_{n-2}), (x_{n-1},f_{n-1}), (x_n,f_n).
Thus:

$$(2.7)\qquad f(x)\big|_{x\in\sigma_{j-1}} \approx P_j(x) := \frac{1}{2h^2}\Big[(x-x_{j-1})(x-x_j)f_{j-2} - 2(x-x_{j-2})(x-x_j)f_{j-1} + (x-x_{j-2})(x-x_{j-1})f_j\Big],$$

$$j = 2,\dots,N,$$

and

$$f(x)\big|_{x\in\sigma_0} \approx P_2(x).$$

Substitution in (1.6) and restricting x to $\{ih,\ i = 1,\dots,N\}$ gives (after a change of variable):

$$(2.8)\qquad g(x_k) = h^{\frac{1}{2}} \sum_{j=0}^{k-1} \int_0^1 \frac{K(x_k,(j+\tau)h)P_f((j+\tau)h)}{(k-j-\tau)^{\frac{1}{2}}}\, d\tau, \qquad k = 1,\dots,N,$$

$$P_f((j+\tau)h) = \begin{cases} P_{j+1}((j+\tau)h), & j = 1,2,\dots,N-1, \\ P_2\ ((j+\tau)h), & j = 0. \end{cases}$$

The integrals are now calculated using 2-point weighted Gauss quadrature;

(2.9) $$\int_0^1 \frac{G(\tau)}{(\ell-\tau)^{\frac{1}{2}}}\, d\tau = \omega_\ell^{(1)} G(a_\ell^{(1)}) + \omega_\ell^{(2)} G(a_\ell^{(2)}) + R_\ell[G(\tau)], \quad \ell = 1,\dots,N.$$

$\omega_\ell^{(1)}, \omega_\ell^{(2)}, a_\ell^{(1)}, a_\ell^{(2)}$ are determined by requiring:

(2.10) $$a_\ell^{(2)} = 1 \quad \text{and} \quad R_\ell[G(\tau)] = 0 \quad \text{for} \quad G(\tau) = \tau^i, \quad i = 0,1,2.$$

This gives a scheme, similar to, but a bit more complicated than (2.5).

3. CO AND COS

These collocation methods have not been published in the literature before, but are analogues of methods for second kind equations, developed by te Riele [5].

COS is a special version of CO, designed to deal with non-smooth solutions of the form:

(3.1) $$f(x) = \psi(x) + x^{\frac{1}{2}}\chi(x), \qquad \psi \text{ and } \chi \text{ smooth.}$$

We describe COS; CO follows immediately from it. Let m be some fixed positive integer. We again introduce the grid (1.5) and put:

(3.2) $$f(x) \approx u(x), \quad u(x)\big|_{x\in\sigma_k} = \sum_{\ell=0}^{m} a_{k\ell}\, \phi_{k\ell}(x), \qquad k = 0,\dots,N-1,$$

where

$$\sigma_k := (x_k, x_{k+1}], \quad k = 1,\dots,N-1,$$

$$\sigma_0 := [x_0, x_1];$$

$$\phi_{k\ell}(x) := [(x-x_k)/h]^\ell, \qquad k = 1,\dots,N-1,$$

$$\phi_{0\ell}(x) := (x/h)^{\ell/2}.$$

Now the coefficients $a_{k\ell}$ have to be determined. Therefore we introduce the so-called collocation parameters:

(3.3) $$0 < \eta_0 < \ldots < \eta_m = 1.$$

We then substitute u(x) for f(x) in (1.6) and restrict x to the set of collocation points

$$\{x_{kj} := x_k + \eta_j h, \quad j = 0,\ldots,m; \quad k = 0,\ldots,N-1\}.$$

After a change of variable we get from (1.6):

(3.4)
$$\sum_{\ell=0}^{m} a_{k\ell} \int_0^{\eta_j} \frac{K(x_{kj},x_k+h\tau)}{(\eta_j-\tau)^{\frac{1}{2}}} \tau^{\ell} d\tau =$$
$$g(x_{kj})h^{-\frac{1}{2}} - \sum_{i=1}^{k-1} \sum_{\ell=0}^{m} a_{i\ell} \int_0^1 \frac{K(x_{kj},x_i+h\tau)}{(k-i+\eta_j-\tau)^{\frac{1}{2}}} \tau^{\ell} d\tau$$
$$- \sum_{\ell=0}^{m} a_{0\ell} \int_0^1 \frac{K(x_{kj},h\tau)}{(k+\eta_j-\tau)^{\frac{1}{2}}} \tau^{\ell/2} d\tau,$$
$$j = 0,\ldots,m, \quad k = 1,\ldots,N-1.$$

(3.5) $$\sum_{\ell=0}^{m} a_{0\ell} \int_0^{\eta_j} \frac{K(x_{0j},h\tau)}{(\eta_j-\tau)^{\frac{1}{2}}} \tau^{\ell/2} d\tau = g(x_{0j})h^{-\frac{1}{2}}, \quad j = 0,\ldots,m.$$

In matrix notation:

(3.6)
$$M_k \vec{a}_k = h^{-\frac{1}{2}}\vec{g}_k - \sum_{i=1}^{k-1} N_{k-i} \vec{a}_i - N_k \vec{a}_0, \qquad k = 1,\ldots,N-1,$$
$$M_0 \vec{a}_0 = h^{-\frac{1}{2}}\vec{g}_0.$$

After substituting $s := \tau/\eta_j$ in the integrals in M_k we calculate these integrals by using m+1-point Gauss quadrature with weight function $(1-s)^{-\frac{1}{2}}$, where the last abscissa is *prescribed* to be equal to one. We thus get

(3.7) $$\int_0^1 \frac{G(s)}{(1-s)^{\frac{1}{2}}} ds = \sum_{i=0}^{m} \omega_i G(\delta_i) + R_k[G(s)],$$

and require:

$$R_k[G(s)] = 0 \quad \text{for} \quad G(s) = \begin{cases} s^i, & k = 1,\dots,N-1, \\ s^{i/2}, & k = 0, \end{cases} \qquad i = 0,\dots,2m.$$

We then choose our collocation parameters η_i in (3.3) to be *equal* to the resulting Gauss abscissae δ_i (with $\delta_m = 1$). Note that the collocation parameters are the same on each interval, with the exception of the first interval.

The integrals in the matrices N_{k-i} are calculated in a similar manner using r-point weighted quadrature with weight functions $(k-i+\eta_j-\tau)^{-\frac{1}{2}}$, $k-i = 1,\dots,N-1$, $j = 0,\dots,m$. Here, we do *not* prescribe any abscissae and again require exactness for $G(s) = s^{i/2}$ on σ_0 resp. for $G(s) = s^i$ on $\sigma_{>0}$, for $i = 0,\dots,2r-1$. To obtain sufficient precision, r must satisfy: $2r \geq m+1$ (see [6]).

The method CO is similar to COS with the exception that the integral on the *first* interval σ_0 is treated in the *same way* as the integrals on other intervals.

REMARKS.

(i) no starting value is required;

(ii) existence of a solution of (3.6) is easy to prove, under the assumption $K(x,t) \neq 0$ for $t \in [x-h,x]$, see the appendix;

(iii) in order to calculate the weights, we need values of the integrals:

$$\int_0^1 \frac{\tau^i}{(\ell-\tau)^{\frac{1}{2}}}\, d\tau \quad \text{and} \quad \int_0^1 \frac{\tau^{i/2}}{(\ell-\tau)^{\frac{1}{2}}}\, d\tau$$

for which we refer to the appendix;

(iv) in order to calculate the integrals in the N_{k-i} in (3.6), we must calculate (and store) $r\times(m+1)\times(N-1)$ weights and abscissae;

(v) CO and COS require the same number of arithmetic operations.

4. PRODUCT INTEGRATION METHODS

These methods were studied by ANDERSSEN et al. [3]. See also [7]. Choose two sets of parameters:

(4.1) collocation parameters $Q := \{0 \le \eta_0 \le \ldots \le \eta_m \le 1\}$

(4.2) evaluation parameters $X := \{0 \le \mu_0 \le \ldots \le \mu_m \le 1\}$

and define

(4.3) $x_{kj} := x_k + \eta_j h$ (collocation *points*)

(4.4) $x^*_{kj} := x_k + \mu_j h$ (evaluation *points*).

We now approximate the function $K(x_{kj},t)\cdot f(t)$ on each interval $\sigma_i = (x_i,x_{i+1}]$ by

$$(4.5)\qquad K(x_{kj},t)f(t)\Big|_{\sigma_i} \approx \sum_{\ell=0}^{m} K(x_{kj},x^*_{i\ell})f_{i\ell}L_\ell\left(\frac{t-x_i}{h}\right)$$

with

$$L_\ell(z) := \prod_{\substack{p=0\\ p\neq\ell}}^{m} \frac{z-\mu_p}{\mu_\ell-\mu_p},$$

and $f_{i\ell}$ is a numerical approximation to $f(x^*_{i\ell})$.

Substitution in (1.6) and some manipulations then yields the scheme:

$$(4.6)\qquad \sum_{\ell=0}^{m} K(x_{kj},x^*_{k\ell})f_{k\ell}\int_0^{\eta_i} \frac{L_\ell(s)}{(\eta_j-s)^{\frac{1}{2}}}\,ds =$$

$$= h^{-\frac{1}{2}}g(x_{kj}) - \sum_{i=0}^{k-1}\sum_{\ell=0}^{m} K(x_{kj},x^*_{i\ell})f_{i\ell}\int_0^1 \frac{L_\ell(s)}{(k-i+\eta-s)^{\frac{1}{2}}}\,ds,$$

$$j = \begin{cases} 1,\ldots,m, & \text{if } \eta_0 = 0,\\ 0,\ldots,m, & \text{if } \eta_0 > 0,\end{cases}\qquad k = 0,\ldots,N-1.$$

REMARKS.

(i) The choice $\mu_0 = \eta_0 = 0$ and $\mu_m = \eta_m = 1$ decreases the dimension of the system (4.6) by one.

(ii) In [3], the authors only consider the case $Q = X$. Brunner [7] proves that, for a certain choice of X, superconverge is obtained in the collocation points x_{kj}.

(iii) Note that in the left-hand part of (4.6) the kernel is evaluated outside its region of definition if $x^*_{k\ell} > x_{kj}$.

5. THE GLOBAL METHOD OF CHAWLA AND KUMAR

For ease of notation we alter the integration bounds in (1.1):

$$(5.1) \qquad \int_{-1}^{x} \frac{K(x,t)f(t)}{(x-t)^{\frac{1}{2}}}\,dt = g(x), \qquad x \in [-1,+1].$$

We now assume that f(x) can be approximated by a series of Chebyshev polynomials:

$$(5.2) \qquad f(x) \approx \sum_{j=0}^{N}{}' a_j T_j(x) \quad \text{and, moreover,}$$

$$(5.3) \qquad K(x,t) \approx \sum_{i=0}^{M}{}'' b_i(x) T_i(t)$$

(' resp." means that the first, respectively the first and the last term are to be halved).

Using the "classical" abscissae:

$$(5.4) \qquad x_k = \cos((2k+1)\pi/(2N+2)), \qquad k = 0,\ldots,N$$

and

$$(5.5) \qquad x_r^{**} = \cos(r\pi/M), \qquad r = 0,\ldots,M$$

we can discretize (5.1) as follows:

$$(5.6) \qquad \sum_{j=0}^{N}{}' a_j \sum_{i=0}^{M}{}'' b_i(x_k) P_{ij}(x_k) = g(x_k), \qquad k = 0,\ldots,N,$$

with

$$(5.6.1) \qquad P_{ij}(x_k) := \int_{-1}^{x_k} T_i(t) T_j(t)/(x_k-t)^{\frac{1}{2}}, \qquad i = 0,\ldots,M,\ k,j = 0,\ldots,N,$$

$$(5.6.2) \qquad b_i(x_k) = 2/M \sum_{r=0}^{M}{}'' K(x_k,x_r^{**}) T_i(x_r^{**}), \qquad i = 0,\ldots,M.$$

For the details of the calculation of (5.6) and the derivation of (5.6.2) we refer to [4].
The coefficients a_j are calculated from the linear system (5.6).

REMARK. In order to use this method, it is necessary to define the kernel $K(x,t)$ on the entire square $-1 \le x,t \le 1$.

6. NUMERICAL EXPERIMENTS

6.1. Test problems

We will present numerical results obtained with the previously discussed methods on the following test problems. These are illustrative examples from our original (much larger) test set. The conclusions in Section 6.3 are based on the results of the original test set. In the following, $M_\nu^a(z) := M(\nu,a,z)$, the Kummer function (cf.[13],p.504).

I. smooth problems

IA
$$f(x) = \sin \tfrac{1}{2}x, \qquad x \in [0,2],$$
$$K(x,t) = 1,$$
$$g(x) = \frac{(\pi x)^{\frac{1}{2}}\Gamma(1)}{i\Gamma(3/2)}[M_1^{\frac{1}{2}}(ix)-M_1^{\frac{1}{2}}(-ix)],$$
ref.: [11].

IB
$$f(x) = \frac{1}{2\sqrt{2\pi x}}\left[(\frac{1}{x}-1)\exp(-\frac{1}{2x}(1+x)^2)+(\frac{1}{x}+1)\exp(-\frac{1}{2x}(1-x)^2-2)\right], \quad x \in [0,2],$$
$$K(x,t) = \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}(x-t)),$$
$$g(x) = \frac{1}{\sqrt{2\pi x}}\exp(-\frac{1}{2x}(1+x)^2),$$
ref.: [2].

II. non-smooth problems

IIA
$$f(x) = x^{\frac{1}{2}}, \qquad x \in [0,2],$$
$$K(x,t) = 1 + \sin 2x \cos 2t,$$
$$g(x) = \frac{\pi x}{2} + \sin(2x).\frac{(\pi x)^{\frac{1}{2}}\Gamma(1)}{\Gamma(3/2)}[M_{3/2}^2(ix)+M_{3/2}^2(-ix)].$$

IIB $\quad f(x) = x^{3/2}, \quad x \in [0,2],$

$K(x,t) = 1,$

$g(x) = 3\pi x^2/8.$

III <u>rapidly oscillating problems</u>

IIIA $\quad f(x) = \sin 16x, \quad x \in [0,1],$

$K(x,t) = 1,$

$$g(x) = \frac{(\pi x)^{\frac{1}{2}}\Gamma(1)}{2i\Gamma(3/2)}[M_1^{16}(ix) - M_1^{16}(-ix)].$$

6.2. <u>Some preliminary remarks</u>

<u>(i)</u> We use the following "coding":

CO-i, COS-i = CO, COS with m+1 = i, $\quad i = 2,3,4;$

HW-i = product-integration with m = 0, $\quad i = 2,3,4$ and $\mu_0 > 0$;

HW-i∅ = as HW-i but $\mu_0 = \eta_0 = 0$;

CK = the method of Chawla and Kumar.

<u>(ii)</u> Convergence.

With p^{th}-order convergence of an approximation $\tilde{f}(x)$ (found by using a certain method) to $f(x)$, we mean:

$$(6.2.1) \qquad \sup_{[0,X]} |f(x)-\tilde{f}(x)| < Ch^p, \quad h \text{ small enough},$$

for some constant C. When speaking about a p^{th}-order method, we mean that for $f(x) \in C^\infty[0,X]$ (6.2.1) holds. For non-smooth $f(x)$, the *actual* order of convergence of a p^{th}-order method may be less than p. With the exception of BR2 and BR3 [2], for none of the discussed methods a general convergence proof is known. Eggermont [9] and Weiss [10] gave proofs of second order convergence for HW-2∅ and in [7] Brunner claims to have proven convergence of order p for HW-p in the special case:

$$(6.2.2) \qquad \eta_j = \tfrac{1}{2}\{1 + \cos[\frac{(2(p-j)+1)\pi}{2p+3}]\}, \quad j = 0,\dots,p,$$

but the paper he refers to has not yet appeared.

Experimentally, we found for CO-i, COS-i, HW-i and HW-i∅ convergence of order i (for f(x) smooth enough). For f(x) of the form (3.1), CO-i, HW-i reduce to second order methods, while COS-i appears to be of order min{i,i/2+3/2} (cf. Section 6.3.2). In [7] Brunner proves for HW-i, that for the choice of Q (cf. Section 4) according to (6.2.2) and

(6.2.3) $\quad X := \{\text{zeros of } P_i^*(x)\}, \quad P_i^*$ the i-th Legendre polynomial,

we get a local order of $i + \frac{1}{2}$ in the points (4.4) while the global order remains i.

For *non*-singular first kind Volterra equations a sufficient and necessary condition for order m+1 convergence is:

(6.2.4) $$\prod_{i=0}^{m} \frac{\eta_i}{1-\eta_i} < 1 \quad \text{(cf.[8])}.$$

For the Abel equation no equivalent for (6.2.4) is known yet.

(iii) On the next pages we give the results of our tests on the previously mentioned test examples.

In each entry of Tables 2.1-2.5 the upper figure denotes the number of correct digits in the endpoint, defined by:

(6.2.5) $$cd := -\log_{10}(\text{absolute error in endpoint}).$$

The lower figure denotes the run time in seconds of our ALGOL 68 program on a CDC CYBER 750 computer. This figure is, of course, machine and programming language dependent, so it has no *absolute* significance but it indicates the performance of the methods in comparison to one another and the growth in computing time with increasing number of steps.

In Figures 1.1 and 1.2 we give a graphical representation of the results obtained for problems IA and IIA, viz., the number of correct digits (cd) versus the run time in seconds (rt).

(iv) For HW-i and HW-i∅ we chose the sets X and Q to be equal. Other choices did not result in a significant increase of the global precision. The choice indicated in the above remark (ii), however, gave, due to the superconverge, a local increase in precision in the evaluation points, but no global increase.

The collocation parameters we have chosen are listed below:

m+1	η_0	η_1	η_2	η_3
2	0	1	–	–
	$\frac{1}{2}$	1	–	–
3	0	$\frac{1}{2}$	1	–
	$\frac{1}{3}$	$\frac{2}{3}$	1	–
4	0	$\frac{1}{3}$	$\frac{2}{3}$	1
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

Table 1

(v) It turned out that for smooth problems, CO and COS gave nearly the same results. Therefore we have not given the results for CO for each problem.

(vi) If $K \equiv 1$ BR2 and HW-2Ø are identical methods.

Table 2.1

Problem nr: IA $\qquad$ $f(x) = \sin \frac{1}{2}x$

cd in x = 2 $\qquad$ $K(x,t) = 1$

h	2nd order methods					3rd order methods				
	CO-2	COS-2	HW-2	HW-2Ø	BR2	CO-3	COS-3	HW-3	HW-3Ø	BR3
1/10	4.00 .14	4.01 .14	4.17 .19	3.78 .02	3.78 .03	6.67 .41	6.67 .41	6.54 .38	5.86 .24	5.53 .04
1/20	4.60 .44	4.61 .44	4.77 .63	4.37 .05	4.37 .08	7.57 1.32	7.57 1.33	7.45 1.21	6.77 .77	6.44 .13
1/40	5.20 1.46	5.20 1.47	5.38 2.36	4.97 .19	4.97 .29	8.48 4.69	8.48 4.63	8.36 4.33	7.67 2.92	7.35 .47
1/80	5.80 5.40	5.80 5.41	5.98 9.64	5.57 .68	5.57 1.09	9.39 17.37	9.39 17.39	9.26 17.54	8.76 11.98	8.26 1.81
1/160					6.17 4.17					9.16 6.97

h	4th order methods			
	CO-4	COS-4	HW-4	HW-4Ø
1/10	9.02 .65	9.00 .64	8.87 .62	8.47 .43
1/20	10.22 2.01	10.20 2.01	10.07 1.97	9.67 1.37
1/40	11.39 7.08	11.38 6.91	11.27 6.99	10.86 4.72
1/80	13.00 25.80	12.44 25.72	12.31 26.54	11.93 19.37

	CK
n \ m	0
2	3.00 .01
3	5.17 .01
4	5.99 .02
6	8.84 .03
8	12.74 .04

Table 2.2

Problem nr: IB $\quad f(x) = \dfrac{1}{2\sqrt{2\pi x}}\left[\left(\dfrac{1}{x}-1\right)\exp\left(-\dfrac{1}{2x}(1+x)^2\right)+\left(\dfrac{1}{x}+1\right)\exp\left(-\dfrac{1}{2x}(1-x)^2-2\right)\right].$

cd in x = 2 $\quad K(x,t) = \dfrac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}(x-t))$

h	2nd order methods				
	CO-2	COS-2	HW-2	HW-2Ø	BR2
1/10	4.86 .17	4.82 .17	5.42 .21	4.26 .01	4.64 .03
1/20	5.49 .50	5.49 .49	6.12 .71	4.81 .02	5.22 .11
1/40	6.09 1.72	6.09 1.75	6.72 2.66	5.40 .08	5.82 .38
1/80	6.70 6.40	6.69 6.49	7.33 11.11	6.00 .29	6.42 1.48
1/160					7.02 5.72

h	3rd order methods				
	CO-3	COS-3	HW-3	HW-3Ø	BR3
1/10	6.41 .45	6.42 .46	6.59 .41	7.18 .28	4.34 .05
1/20	8.05 1.45	8.26 1.50	7.83 1.36	7.55 .95	6.57 .18
1/40	8.74 5.40	8.74 5.36	9.07 5.15	8.58 3.53	7.63 .67
1/80	9.65 20.24	9.65 20.12	9.97 20.96	9.49 14.36	8.56 2.57
1/160					9.46 10.13

h	4th order methods			
	CO-4	COS-4	HW-4	HW-4Ø
1/10	7.56 .73	6.82 .73	7.11 .69	7.71 .54
1/20	9.43 2.29	9.08 2.30	8.72 2.35	9.16 1.77
1/40	11.17 7.89	11.16 8.04	10.54 8.44	10.69 6.44
1/80	12.42 29.61	12.43 28.17	11.68 32.16	11.79 26.01

	CK		
n \ m	2	4	8
2	1.29 .01		
4	1.42 .02	1.42 .04	
8		1.75 .07	1.75 .16
16			2.49 .43

Table 2.3

Problem nr. IIA $\qquad f(x) = x^{\frac{1}{2}}$

cd in x = 2 $\qquad K(x,t) = 1 + \sin 2x \cos 2t$

h	2nd order methods				
	CO-2	COS-2	HW-2	HW-2∅	BR2
1/10	3.50 .19	3.47 .19	3.38 .23	2.60 .03	2.81 .04
1/20	3.95 .56	3.94 .56	3.97 .76	3.19 .09	3.42 .11
1/40	4.48 1.91	4.47 1.90	4.58 2.87	3.78 .35	4.03 .43
1/80	5.04 6.96	5.03 6.98	5.18 11.82	4.38 1.26	4.63 1.66
1/160					5.24 6.36

h	3rd order methods				
	CO-3	COS-3	HW-3	HW-3∅	BR3
1/10	5.68 .51	6.15 .51	4.85 .45	4.44 .31	3.59 .06
1/20	6.38 1.66	7.60 1.66	5.61 1.51	5.16 1.02	4.75 .21
1/40	7.03 5.82	8.24 5.80	6.28 5.50	5.89 3.87	5.96 .75
1/80	7.66 21.92	8.64 21.89	6.92 22.20	6.57 15.20	6.56 2.92
1/160					

h	4th order methods			
	CO-4	COS-4	HW-4	HW-4∅
1/10	7.07 .77	5.92 .77	5.62 .77	4.94 .58
1/20	7.41 2.46	6.97 2.44	6.19 2.53	5.59 1.89
1/40	7.94 8.50	8.03 8.53	6.79 9.12	6.20 6.87
1/80	8.52 32.02	9.11 32.02	7.50 35.93	6.72 27.23

	CK		
n \ m	2	4	8
2	0 .02		
4	0 .03	1.8 .05	
8		2.1 .07	2.5 .17

Table 2.4

Problem nr: IIB

cd in x = 2

$f(x) = x^{3/2}$

$K(x,t) = 1$

h	2nd order methods					3rd order methods				
	CO-2	COS-2	HW-2	HW-2Ø	BR2	CO-3	COS-3	HW-3	HW-3Ø	BR3
1/10	3.59 .16	3.59 .16	3.77 .17	3.35 .01	3.35 .02	6.44 .42	6.28 .44	6.15 .35	5.76 .25	5.03 .04
1/20	4.19 .46	4.19 .47	4.38 .59	3.96 .04	3.96 .08	7.36 1.35	7.18 1.43	7.05 1.13	6.68 .80	6.16 .13
1/40	4.80 1.52	4.80 1.58	4.98 2.28	4.56 .17	4.56 .28	8.26 4.70	8.09 4.96	7.95 4.13	7.59 2.96	7.07 .46
1/80	5.40 5.63	5.40 5.83	5.59 9.31	5.16 .62	5.16 1.03	9.16 18.12	9.00 18.83	8,86 16.86	8.49 11.84	7.97 1.75
1/160				5.76 2.44	5.76 4.12					

h	4th order methods			
	CO-4	COS-4	HW-4	HW-4Ø
1/10	8.30 .66	8.69 .68	7.76 .60	7.43 .45
1/20	9.28 2.07	9.89 2.14	8.63 1.89	8.38 1.47
1/40	10.25 7.26	11.01 7.37	9.46 6.75	9.33 5.19
1/80	11.17 26.85	12.18 27.74	10.51 28.13	9.43 20.45

CK			
n \ m	0	n \ m	0
2	1.4 .01	16	5.31 .16
3	2.22 .01	24	6.40 .37
4	2.62 .01	32	6.34 .71
6	3.20 .03	48	6.85 1.81
8	3.72 .04		

Table 2.5

Problem nr: IIIA

cd in x = 1

f(x) = sin 16x

K(x,t) = 1

h	2nd order methods					3rd order methods				
	CO-2	COS-2	HW-2	HW-2Ø	BR2	CO-3	COS-3	HW-3	HW-3Ø	BR3
1/10	1.55 .06		1.87 .06	1.46 .01	1.46 .03	1.90 .17		1.85 .13	1.54 .09	0
1/20	2.90 .16		2.78 .17	2.44 .02	2.44 .11	2.78 .45		2.72 .34	2.27 .25	1.72 .14
1/40	2.97 .49		3.00 .56	2.70 .06	2.70 .31	3.72 1.44		3.64 1.07	3.13 .81	2.61 .51
1/80	3.41 1.61		3.50 1.97	3.19 .20	3.20 1.13	5.01 5.06		4.80 3.80	4.09 2.95	3.52 1.83
1/160				3.73 .70	3.76 4.29					4.61 7.22

h	4th order methods			
	CO-4	COS-4	HW-4	HW-4Ø
1/10	3.45 .24		3.33 .22	2.87 .18
1/20	4.84 .70		4.80 .58	4.70 .46
1/40	5.34 2.10		4.79 1.77	4.75 1.45
1/80	6.57 7.33		4.97 6.20	4.81 5.22

CK			
n \ m	0	n \ m	0
2	0	24	7.61 .37
4	0	32	8.04 .66
6	1.00 .03	48	8.87 1.77
8	1.00 .04		
16	2.93 .16		

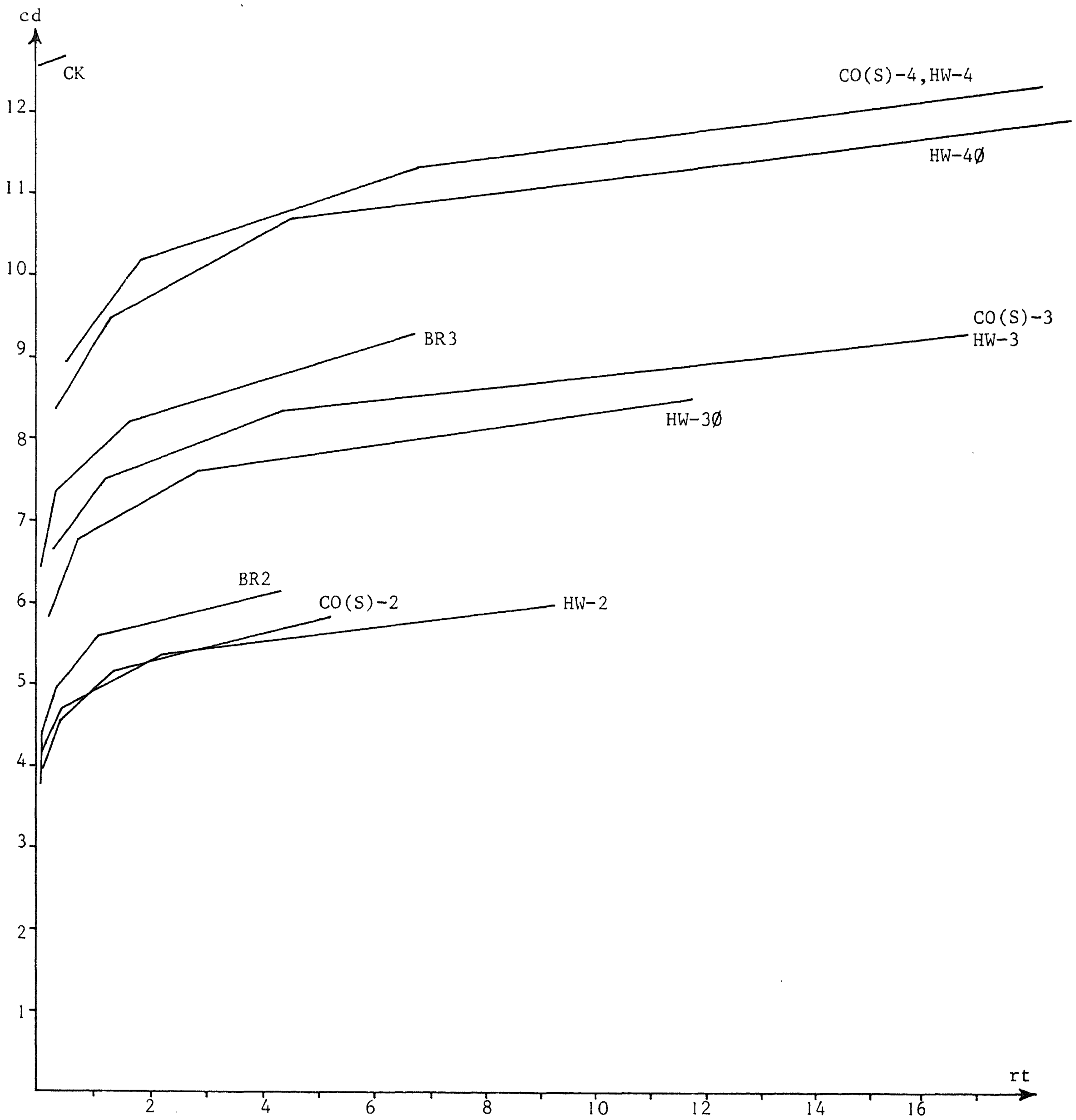

Figure 1.1 Problem IA

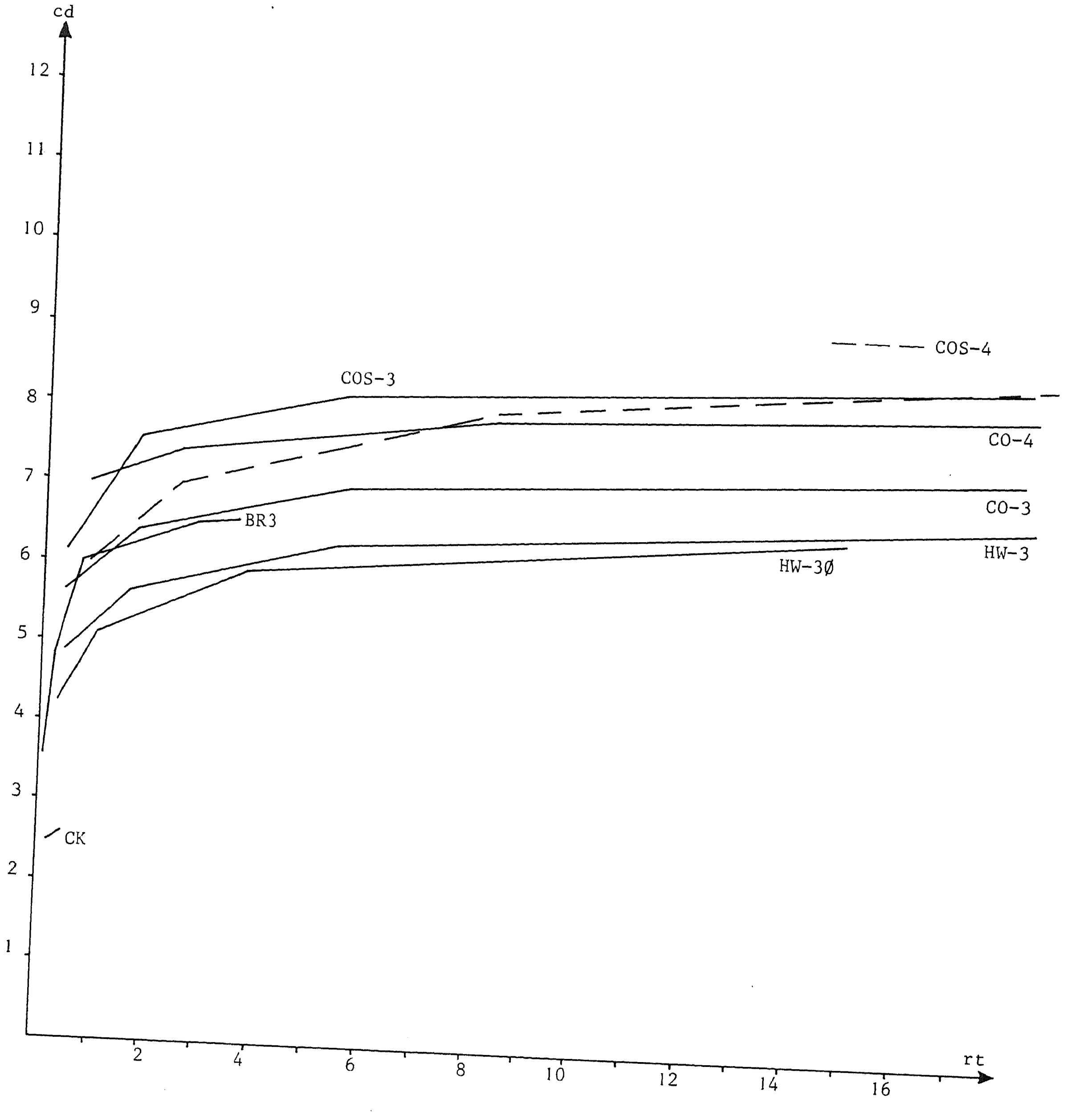

Figure 1.2 Problem IIA

6.3. CONCLUSIONS

6.3.1. Smooth solutions

Striking (cf. problem IA) is the very good behavior of Chawla/Kumar for most of these problems. The attainable results are, however, strongly influenced by the suitability of the kernel and/or solution of being approximated by a polynomial. Compare, e.g., the problems IA and IB. Chawla/Kumar behaves dramatically worse for the latter.

A second remarkable fact is the equal behavior of CO and COS for this class of problems.
The accuracy of CO on the first interval is in general higher than that of COS, but this difference disappears. This is quite contrary to the behavior of the analoguous second-kind equation solvers of te Riele [2]: the counterpart of COS behaves worse for smooth solutions.

Concerning computing time, it is clear that the fact that BR2 and BR3 (and HW2Ø) don't have to solve a system in each step, is a great advantage, but the decrease in accuracy is considerable. Still, BR3 seems to be the most efficient one among the third order methods. It is interesting to note that HW3 and HW4 are not significantly less efficient than HW3Ø and HW4Ø for trivial kernels. The advantage that the dimension of the system to be solved is one smaller, is in general annihilated by the decrease in precision. For more expensive kernels, however, the fact that also the number of kernel evaluations for HW-iØ is much smaller, plays a dominant role (cf. Table 3 in Section 7). This is also the reason why HW is less efficient than CO or COS for non-trivial kernels. The difference in kernel evaluations between CO(S) and HW-iØ is only small, and from our test results it is clear that, at least for orders 3 and 4, HW-iØ is only slightly more efficient than CO(S). Finally, it is clear that no lower order method is more efficient than any higher order one.

6.3.2. Non-smooth solutions

Firstly, we note that the good behavior of Chawla/Kumar does not extend to this class of solutions. This is not surprising, of course, as a non-smooth function is badly approximated by a Chebyshev polynomial.

An important question is how the specially developed method COS behaves. It is remarkable that in the case of $m = r = 1$ COS-2 does not behave significantly better than CO-2, although the approximation on the first interval is much better, in the case of problem IIA even exact. Note that COS-2 and CO-2 both have order 2 for these problems. For higher order methods the advantage of COS becomes clear. We see that for solutions of the form (3.1) with $\chi(0) \neq 0$, all collocation and product integration methods but COS reduce to order 2 methods, while the order of COS-i seems to be $\min\{i+1, \frac{i+1}{2} + \frac{3}{2}\}$. An heuristical explanation for this may be the following: On the first interval, a function like $x^{\frac{1}{2}}$ cannot be approximated by a polynomial with more than $h^{\frac{1}{2}}$ accuracy. In the expression for the error equation, which has a form similar to 3.6 (without $g(x)$), we multiply this approximation with a term that behaves like $h^{3/2}$, $h \to 0$ $(N \to \infty)$. On the other intervals where $x^{\frac{1}{2}}$ is smooth, the approximation is of the order i. So the global order of convergence will be $\min\{1/2+3/2, i\}$ which is always 2. For COS-i, however, the accuracy on the first interval is $h^{i/2}$ (which follows from a Taylor expansion of $\psi(x) + x^{\frac{1}{2}}\chi(x)$ near 0). So the global order of convergence will be $\min\{i/2+3/2, i\}$, which is i for i = 2,3 and $3\frac{1}{2}$ for i = 4. This order of $3\frac{1}{2}$ is detected in problem IIA.

If $\chi(0) = 0$ but $\chi^{(p)}(0) \neq 0$, for some $p > 0$, a similar reasoning could be held, but all this is not mathematically founded, as no convergence proofs are known to us, not even for smooth problems.

Nevertheless, it will be clear that the idea of fitting the method to the solution pays.

6.3.3. Rapidly decreasing or oscillating solutions

We can be quite short on these problems. Of course the accuracy of all methods is decreased but the results remain acceptable. For oscillating problems, Chawla/Kumar also behaves rather good (provided the degree is high enough) under the same restrictions as for ordinary smooth problems. There is no change in the relative order of the methods.

6.3.4. Concluding remarks

To solve equation (1.1), product integration - and collocation methods are reliable. If enough about the solution is known, one might consider using Chawla/Kumar (if the solution is smooth). The facts that (a) COS is not inferior to CO for smooth problems, (b) HW-iØ behaves only slightly better than CO (and COS) for smooth problems, (c) COS-3,4 are superior to all other methods for non-smooth problems, may lead to the conclusion that, when high accuracy is requested, COS-4 is the most reliable choice. It can handle both smooth and non-smooth solutions. If only a relatively low accuracy is required, a good choice would be BR2 or BR3, and these methods have the additional advantage that the implementation is easier because no systems have to be solved.

7. COMPUTING TIME AS A FUNCTION OF THE NUMBER OF CORRECT DIGITS

For the product-integration and collocation methods, it is possible to compare the results in a rather unified manner. Therefore we define:

(6.3.1) $W_q(N)$:= time required by method q to take N steps without accounting for kernel evaluations.

Then $W_q(N)$ is (almost) problem independent and it is obvious that $W_q(N)$ will be quadratic in N. So:

$$W_q(N) = c_0^{(q)} + c_1^{(q)}N + c_2^{(q)}N^2, \tag{6.3.2}$$

with $c_0^{(q)}, c_1^{(q)}$ and $c_2^{(q)}$ method-dependent coefficients. Furthermore we define:

(6.3.3) $P(N)$:= the number of correct digits in X calculated in N steps (or: with stepsize $h = X/N$)

then $P(N) = a + m \log_{10}(N)$,

where a is some problem and method dependent constant, and m the order of the method (which depends on the smoothness of the solution). The total time needed is:

$$W_q^*(N) = W_q(N) + d^{(q)}(N) \cdot T. \tag{6.3.4}$$

T is the time required for one kernel evaluation and $d^{(q)}(N)$ is the total number of kernel evaluations which is quadratic in N and problem-independent. We list $d^{(q)}(N)$ for the various methods below.

Table 3

	order 2	order 3	order 4
BR	$\frac{1}{2}N^2+\frac{1}{2}N$	N^2+N	-
CO(S)	N^2+2N	$3N^2+6N$	$4N^2+12N$
HW	$2N^2+2N$	$\frac{9}{2}N^2+\frac{9}{2}N$	$8N^2+8N$
HW-iØ	$\frac{1}{2}N^2+\frac{1}{2}N$	$2N^2+2N$	$\frac{9}{2}N^2+\frac{9}{2}N$

For (6.3.3) we can write:

$$(6.3.5)\qquad N = 10^{(P-a)/m}.$$

Inserting (6.3.5) in (6.3.4) yields:

$$(6.3.6)\qquad W_q^*(P) = c_0^{(q)} + c_1^{(q)}\,10^{(P-a)/m} + c_2^{(q)}\,10^{2(P-a)/m} + d^{(q)}(10^{(P-a)/m})\cdot T.$$

(6.3.6) gives an expression for the computing time as a function of the required number of correct digits P.

The $c_0^{(q)}$, $c_1^{(q)}$ and $c_2^{(q)}$ can be estimated from our test results if we take some problem with $K \equiv 1$ (which means $T = 0$). The parameter a, which is problem dependent, can also be found from the test results.

We give two examples:

Table 4

method	c_0	c_1	c_2	problem	a	m
HW4	4.10^{-2}	8.5×10^{-3}	9.5×10^{-4}	IA	3.65	4
				IIA	2.31	2
HW4Ø	8.10^{-2}	4×10^{-3}	7.5×10^{-4}	IA	3.24	4
				IIA	2.40	2

The problem is that it is often difficult to determine the value of m. (cf. problem IB)

APPENDIX

A frequent use has been made of values of the integrals

(A1) $$\int_0^1 \frac{\tau^{\nu}}{(x-\tau)^{\alpha}}\, d\tau \quad \text{for} \quad \alpha = \tfrac{1}{2}.$$

These can be calculated from the recurrence relations:

(A2) $$J_{0,\alpha}(x) := \frac{1}{1-\alpha}[x^{1-\alpha}-(x-1)^{1-\alpha}]$$

(A3) $$J_{r,\alpha}(x) = \frac{\nu}{1-\alpha} J_{\nu-1,\alpha-1}(x) - \frac{(x-1)^{1-\alpha}}{1-\alpha} .$$

From this it is easy to derive that:

(A4) $$\int_0^{\eta_j} \frac{\tau^{\nu}}{(\eta_j-\tau)^{\frac{1}{2}}}\, d\tau = \frac{2^{\nu+1}.\nu!}{1\cdot 3\cdot \ldots \cdot(2\nu+1)}\, \eta_j^{\nu+\frac{1}{2}} .$$

Using relation (A4), the fact that $K(x,t) \neq 0$, $t \in [x-h,x]$, and the mean-value theorem for integration, it is easy to prove the independence of the columns of M_k in (3.6), hence (3.6) has a unique solution.

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